SELF-SIMILARITY OF FRICTION LAWS

Maria de Sousa Vieira and Hans J. Herrmann

Hochstleistungsrechenzentrum Supercomputing Center, Kernforschungsanlage
D-52425 Jülich
Germany

ABSTRACT

The change of the friction law from a mesoscopic level to a macroscopic level is studied in the spring-block models introduced by Burridge-Knopoff. We find that the Coulomb law is always scale invariant. Other proposed scaling laws are only invariant under certain conditions.

The first documented studies on friction were done by Leonardo da Vinci, who experimentally verified two basic laws of friction. His studies were rediscovered by Amonton de la Hire who announced the two laws in 1699 in the following form (a) the friction force is independent of the size of the surfaces in contact, and (b) friction is proportional to the normal load. Nearly 100 years after Amonton, Coulomb recognized the difference between static and dynamic friction. He noticed that the initial friction increased with the time the surfaces were left in stationary contact. During the slip process the friction is smaller than the static friction, and he proposed the dynamic friction to be constant (For these accounts see for example Cap. 2 of Ref. [1]).

More recent experimental studies on friction show, however, that friction does vary during sliding, decreasing with slip[2,3]. This is called *slip weakening*. In particular, if the friction has a negative dependence on the sliding velocity it is called *velocity weakening*. In these cases a dynamic instability can occur, resulting in very sudden skip with an associated stress drop. This often occurs repetitively - a period of rest follows the slip, which in turn is followed by a period of rest, and so on. This behavior is called stick slip motion. It is commonly observed in the frictional sliding of rocks, which led Brace and Byerlee[4] to propose it as a mechanism for earthquakes. Besides the stick slip process, two surfaces with friction can also have a steady relative motion, with the sliding velocity constantly positive.

In an experimental investigation one understands friction force as a macroscopic average of the resistance to motion due to microscopic interactions between two sliding surfaces. The microscopic forces have various sources and they are a topic of great interest nowadays. Several theoretical models for friction have been proposed recently[5,6]. In these models one considers a mesoscopic level and studies how individual elements (having a given mesoscopic friction behavior) synchronize to a collective behavior giving the macroscopic phenomena like the classical Coulomb law or the stick slip motion. The question arises - Will these mesoscopic friction laws reproduce the macroscopic behavior observed experimentally, and are they consistent with the phenomenological models that have been proposed? This fundamental question is the subject of this paper.

We investigate two mechanical models introduced by Burridge and Knopoff[7] to mimic the dynamics of earthquakes. Each model consists of a chain of blocks connected by springs, the set being driven at constant velocity on a surface with friction. Several studies of these models have been performed showing that, at least qualitatively, that they can present the dynamics observed in real earthquakes [7-10]. We study the relation between the macroscopic and mesoscopic behavior of friction using four different friction laws, which are introduced here in the same chronological order they appeared in the literature.

The first model is shown schematically in Fig. 1(a). It consists of a chain of N blocks of mass m coupled to each other by harmonic springs of strength k. The blocks are on a

surface and the first one is pulled with constant velocity v. Between the blocks and the surface acts a velocity dependent frictional force f, which is usually nonlinear, giving the system a rich behavior. This model has been called "the train model"[9] to distinguish it from the other model, which was also introduced by Burridge and Knopoff.

The equation of motion for a moving block j is given by

$$m\ddot{X}_j = k(X_{j+1} - 2X_j + X_{j-1}) - f(\dot{X}_j/v_c), \quad \dot{X}_j \neq 0, \tag{1}$$

with $j = 1, \ldots, N$. X_j denotes the displacement of the j-th block measured with respect to the position where the sum of the elastic forces in the block is zero. The open boundaries are given by $X_{N+1} = X_N$ and $X_0 = vt$. In Eq. (1) we consider the friction force as function of the instantaneous block velocity with respect to a characteristic velocity v_c. If we write $f(\dot{X}/v_c) = f_\circ \Phi(\dot{X}/v_c)$ where $\Phi(0) = 1$ and introduce the variables $\tau = \omega_p t$, $\omega_p^2 = k/m$, $U_j = kX_j/f_\circ$, Eq. (1) can be written in the following dimensionless form[9]

$$\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} - \Phi(\dot{U}_j/\nu_c), \quad \dot{U}_j \neq 0, \tag{2}$$

with $U_{N+1} = U_N$, $U_0 = \nu\tau$, $\nu = v/V_\circ$, $\nu_c = v_c/V_\circ$, and $V_\circ = f_\circ/\sqrt{km}$. Dots now denote differentiation with respect to τ. In a system of a single block the quantity $f_\circ/\omega_p$ is the maximum displacement of the pulling spring before the block starts to move; in the absence of dynamical friction $2\pi/\omega_p$ and $V_\circ$ are respectively a characteristic period of oscillation of the block and the maximum velocity it attains. This system is completely described by two dimensionless parameters, ν and ν_c.

The other model we study is shown schematically in Fig. 1(b). It consists of N identical blocks of mass m connected to each other through linear springs of constant k_c. The system is driven by an upper bar moving at constant velocity v with respect to the lower supporting surface at rest. Each mass is attached to the upper bar by a linear spring of constant k_p. Between the masses and the supporting surface there is again a velocity dependent friction force f. We will call this system the BK model.

The equation of motion for a moving block j is given by

$$m\ddot{X}_j = k_c(X_{j+1} - 2X_j + X_{j-1}) - k_p(X_j - vt) - f(\dot{X}_j/v_c), \quad \dot{X}_j \neq 0, \tag{3}$$

with $j = 1, \ldots, N$. We use open boundary conditions, which are given by $X_0 = X_1$ and $X_{N+1} = X_N$. X_j denotes the displacement of the j-th block measured from the point where the sum of all the elastic forces in the block is zero.

Normalizing the parameters and variables in a similar way as above (where k is replaced by k_p) the BK model has the following equation of motion[8]

$$\ddot{U}_j = l^2(U_{j+1} - 2U_j + U_{j-1}) - U_j + \nu\tau - \Phi(\dot{U}_j/\nu_c), \quad \dot{U}_j \neq 0, \tag{4}$$

3

with $l^2 = k_c/k_p$. l can be interpreted as the velocity of the sound in the chain. The system has therefore three fundamental parameters, namely, l, ν and ν_c.

The friction force Φ models the mesoscopic interaction between the blocks and the surface. We call it the *mesoscopic* friction force. The *macroscopic* friction force, denoted here by $\overline{F}$, is given by a temporal average of the force applied by the driving mechanism normalized to the number of blocks. For the train and the BK models the applied forces are given respectively by

$$F_t = \nu\tau - U_1, \tag{5}$$

$$F_{BK} = \nu\tau - \sum_{j=1,N} U_j. \tag{6}$$

Thus, $\overline{F} = <F>/N$, where $<...>$ stands for temporal average. The normalized friction force is in fact what is called the friction coefficient.

Unless otherwise stated, in the numerical simulations we start the system with all the blocks at rest. In the train model the initial position for each block is taken $U_j = 0$, i.e., at equal distances. Even with a perfectly homogeneous initial configurations a complex dynamics may naturally emerge in the train model[9]. In the BK model a small inhomogeneity in the positions of the blocks must be introduced, otherwise the system will not present any complex behavior, and the chain will move as a single block[8]. For computational convenience we do not allow backward motion of the blocks, that is, the friction force will attain a sufficiently high value in order to forbid backward motion. Tests have been performed where backward motion was allowed and we did not observe any significant difference from the results shown here. In our calculations we use open boundary conditions for the chain. Several tests have shown that our results do not change if periodic boundary conditions are used.

We have studied for Φ four different functions found in literature to verify which ones will conserve their functional form when going from the mesoscopic to the macroscopic level in the mechanical models introduced by Burridge and Knopoff. Since scale invariance is often observed in the roughness of solid surfaces, a functional similarity of the friction law seems to be a condition that should be fulfilled in order that this law can be applied in these cases.

(a) Coulomb force

The friction law formulated by Coulomb is independent of the value of the sliding velocity and is a discontinuous function at zero velocity given by

$$\Phi(\dot{U}/\nu_c) = \begin{cases} 1, & \text{if } \dot{U} = 0; \\ \Phi_\circ, & \text{if } \dot{U} > 0, \end{cases} \tag{7}$$

4

where $\Phi_\circ < 1$. Note that using this friction force the only nonlinearity in the system is this discontinuity in Φ.

To clarify the dynamics of the mechanical models using Coulomb force, we start by investigating the dynamical evolution of a one-block system. The equation of motion for this system in dimensionless quantities is

$$\ddot{U} = -U + \nu\tau - \Phi_\circ, \quad \dot{U} > 0. \tag{8}$$

A possible solution for this equation is $U^e = \nu\tau - \Phi_\circ$, which gives $\dot{U}^e = \nu$. That is, the block is moving with constant velocity, equal to the pulling speed. The superscript e denotes equilibrium position. The stability of this solution can by investigated by perturbing it. If one writes $U = U^e + u\exp(\omega\tau)$ and substitutes this equation into Eq. (8), one will find $|\omega| = 1$, which implies that the solution with the block having constant velocity is marginally stable. A nontrivial solution is

$$U = (\nu_o - \nu)\sin\tau + (U_o + \Phi_\circ)\cos\tau + \nu\tau - \Phi_\circ, \tag{9}$$

where U_o and ν_o denote $U(\tau = 0)$ and $\dot{U}(\tau = 0)$, respectively. If $\dot{U} = 0$ and the elastic force is smaller than the static friction, the block will stick to the surface until the elastic force becomes again larger than the static frictional force. The evolution of the system in in phase space is shown in Fig. 2(a) for a given example. It consists of concentric ellipses around the trivial solution ($U = U^e$ and $\dot{U} = \dot{U}^e$) when the initial conditions are such that $(\nu_o - \nu)^2 + (U_o + \Phi_\circ)^2 < \nu^2$. If this condition is not satisfied, then the block will stick to the surface, and harmonic motion will occur only in the upper region of the phase space, where $\dot{U} > 0$. The trajectories in phase space evolve symmetrically around the trivial solution, which allows us to conclude that the average elastic force is given by $\overline{F} = \Phi_\circ$ in both cases, that is, when the motion is steady or via stick slip.

For the train model with $N > 1$, the trivial situation in which all the blocks move with constant velocity, equal to the pulling velocity, has $\ddot{U}_j = 0$ for each j. From Eq. (2) we find

$$U_{j+1}^e - 2U_j^e + U_{j-1}^e = \Phi_\circ, \quad j = 1, ..., N \tag{10}$$

with $U_0^e = \nu\tau$ and $U_{N+1}^e = U_N^e$. The superscript e in U denotes again equilibrium position. For the first block, $j = 1$, the equilibrium position is easily found. If one adds the equations given by Eq. (10) with $j = 1, ..., N$, one will find $U_1^e = \nu\tau - N\Phi_\circ$. This gives $\overline{F}_t = \Phi_\circ$. The equilibrium position for any other block is a function of the system size, and is found by solving the system of equations given by Eq. (10).

In situations where no block of the chain sticks to the surface (this occurs for example when $\Phi_\circ = 1$), the motion of the first block is governed by

$$\ddot{U}_1 = -\Omega^2 U_1 + \nu\tau/N - \Phi_\circ, \tag{11}$$

where $\Omega^2 = 1/N$. The solution of Eq. (11) is

$$U_1 = \frac{1}{\Omega}(\nu_o - \nu)\sin(\Omega\tau) + (U_o + N\Phi_\circ)\cos(\Omega\tau) + \nu\tau - N\Phi_\circ, \qquad (12)$$

where U_o and ν_o are the position and velocity of the first block at $\tau = 0$. The motion of the first block is harmonic and it occurs in phase space around the equilibrium solution $U_1 = U_1^e$ and $\dot{U}_1 = \dot{U}_1^e = \nu$. The average value of U_1 obtained from Eq. (12) is $\overline{U}_1 = \nu\tau - N\Phi_\circ$. If we use this result in Eq. (5) we find $\overline{F}_t = \Phi_\circ$.

When $\Phi_\circ < 1$ the blocks can in principle stick to the surface. In this case, the evolution of the system becomes nonlinear, and the integration of the equations of motion does not seem to be very simple as for a one-block system. However, even when sticking occurs in a system with more than one block we find numerically that $\overline{F}_t = \Phi_\circ$. In all investigated cases we found that the first block always has a symmetric trajectory around the trivial solution, whether or not blocks stick to the surface. As a simple example, we discuss a two-block system ($N = 2$). Take $\Phi_\circ = 0.8$ and the initial positions and velocities given respectively by $U_1 = U_2 = 0$, $\dot{U}_1 = \dot{U}_2 = 0$. If $\nu = 0.1$ we see that the first block sticks to the surface and executes a period two orbit around the trivial solution $U_1^e = \nu\tau - 2\Phi_\circ$. If the pulling velocity is increased to $\nu = 1$, the motion of the first block becomes more complicated, and executes what seems to be a quasiperiodic orbit, but also symmetric around the trivial solution. These two different motions are shown in Fig. 2(b) and 2(c), respectively. For larger systems the orbits described by the blocks in phase space become more and complicated. However, the first block seems always to have a symmetric trajectory around the trivial solution, which from Eq. (5) results in $\overline{F}_t = \Phi_\circ$.

For the BK model, shown in Fig. 1(b), the macroscopic force can also be written as

$$F_{BK} = \nu\tau - NW, \qquad (13)$$

where W is the coordinate of the center-of-mass, defined as

$$W = \frac{1}{n}\sum_{j=1,N} U_j. \qquad (14)$$

Thus, for this model is the evolution of W, and not of U_1, that will determine the the value of the macroscopic force.

The evolution for W when no block sticks to the surface is found by adding Eq. (4) with $j = 1, ..., N$. The following expression is obtained

$$\ddot{W} = -W + \nu\tau - \Phi_\circ. \qquad (15)$$

The trivial solution in which all blocks move with the same velocity has $\ddot{U}_j = 0$, for all j. This results in $W^e = \nu\tau - \Phi_\circ$. Using this result in Eq. (14) we find for this trivial

motion $\overline{F}_{BK} = \Phi_\circ$. A simple linear stability analysis shows that this solution is marginally stable. A nontrivial solution for Eq. (15) is

$$W = (\nu_o - \nu)\sin\tau + (W_o + \Phi_\circ)\cos\tau + \nu\tau - \Phi_\circ, \qquad (16)$$

where W_o and ν_o denote $W(\tau = 0)$ and $\dot{W}(\tau = 0)$. The motion for W is harmonic and it evolves around $W = W^e$ and $\dot{W} = \dot{W}^e$, which gives again $\overline{F}_{BK} = \Phi_\circ$.

These results show that when sticking does not occur (this is the case, for example, when $\Phi_\circ = 1$) the motion of center-of-mass of the chain has an equation completely identical to the equation of the one-block system (Eq. (8)). When the nonlinearity in Φ is present we have noticed that the coupling between the blocks strongly affects the dynamics of the system. To exemplify this we show the temporal evolution of a BK system with and without coupling when it is governed by the Coulomb law. Fig. 3 consists of diagrams of the block number j versus the time τ with a dot representing $\dot{U}_j > 0$. For Fig. 3(a) the coupling between the blocks is zero, that is, $l = 0$. The starting conditions are such that $\dot{U}_j(\tau = 0) = 0$ and $U_j(\tau = 0)$ is randomly distributed between $[-0.01, 0.01]$. The parameters are $\Phi_\circ = 0.8$, $\nu = 0.01$ and $N = 100$. We see that the blocks periodically stick to the surface, as also seen in Fig. 2(a). When the coupling is added between the blocks the behavior becomes very different, as Fig. 3(b) shows. There we have taken $l^2 = 50$. We see pulses traveling through the chain with the sound velocity, and the white regions have smaller area than in the case where the coupling is zero. A transient time has been discarded in all simulations shown in this paper. For the train model we start our computations after the last block has moved. For the BK model we discard the time corresponding to ten loading periods, where the loading period is given by $1/\nu$.

We have investigated numerically the evolution of W when sticking to the surface is in principle possible. We observed that a plot of $\dot{W}$ versus $W - \nu\tau$ is qualitative similar to trajectories shown in Fig. 2(a). This results in $\overline{F}_{BK} = \Phi_\circ$.

We have also performed numerical simulations in larger chains. In figure 4(a) and 4(b) we show temporal evolutions for F_t and F_{BK} using Coulomb law, with $N = 200$ for the train model and $N = 500$ for the BK model. The other parameter values are $\nu = 5$, $\Phi_\circ = 0.8$ and $l^2 = 50$. In Fig. 4(a) we see that the evolution of U_1, although complex, seems to be periodic. For the BK model, the evolution of F is also periodic, and has a much simpler structure. The frequency of F_{BK} is one, which is the natural frequency associated with the center-of-mass motion, as found from equation (15). In both cases we find $\overline{F} = \Phi_\circ$.

We have varied the pulling velocity ν and the numerical values found for the macroscopic force in the two models are shown in Fig. 4(c). The parameters used are the same as in Figs. 4(a) and 4(b), with exception of ν, which now is varying. We represent the mesoscopic friction force Φ by the solid line, the macroscopic force $\overline{F}$ by circles for the

train model and by squares for the BK model. There is an excellent agreement between the circles and the squares with the solid line. The small deviations are within the statistical errors. Therefore, we have observed that the Coulomb law gives the same behavior for the mesoscopic force as for the macroscopic force in both models. In other words, our results show that the Coulomb friction force is scaling-invariant and the procedure can be iterated in a renormalization approach.

(b) Dietrich-Scholz friction force

In experimental studies on rock friction, Dietrich[2] and Scholz *et al.*[3] have found that the friction force has a logarithmic dependence on the sliding velocity, decreasing as the velocity increases. They proposed a friction force of the form

$$\Phi(\dot{U}/\nu_c) = \Phi_\circ - b\log(\dot{U}/\nu_c). \tag{17}$$

It is clear that, due to the logarithmic dependence, cutoffs have to be introduced to this function for high and low velocities. We introduce cutoffs for small and large velocities in such a way that the function Φ remains continuous. We consider, then

$$\Phi(\dot{U}/\nu_c) = \begin{cases} 1, & \text{if } \dot{U} < \nu_c; \\ 1 - b\log(\dot{U}/\nu_c), & \text{if } \nu_c \leq \dot{U} \leq \nu_c\exp(1/b); \\ 0, & \text{if } \dot{U} > \nu_c\exp(1/b). \end{cases} \tag{18}$$

For small pulling velocities, in both models, the chain experiences only the linear part of the friction force with $\dot{U} \leq \nu_c$, where we have $\Phi_\circ = 1$. In this case, stick of blocks to the surface does not occur and analytical results can be found for $\overline{F}_t$ and $\overline{F}_{BK}$, as shown in the previous subsection.

The nonlinear regime is attained when the maximum velocity attained by a block is equal or greater than ν_c. For the train model we have to consider two distinct situations. When we start the system with all the blocks at rest, a simple linear analysis, similar to the one performed in subsection (a), shows that the maximum velocity attained by the blocks is equal to ν, if the slipping event does not involve all the blocks of the system. If all the blocks are involved in the event, then the maximum velocity of the blocks is 2ν. This can easily be found from Eq. (12). Thus, when $\nu < \nu_c/2$ the chain never feels the the nonlinear part of Eq. (18) and moves continuously. This results in $\overline{F}_t = \Phi_\circ = 1$. When $\nu_c/2 \leq \nu \leq \nu_c$ the events that do not involve all the blocks of the chain will not feel the nonlinear regime, whereas the events where all the blocks of the chain are displaced will be in the nonlinear regime of the friction force. In this situation, as time evolves F_t increases monotonically until the last block moves, then we see a sharp drop in F_t. Then F_t starts to increase monotonically again, and the process repeats. For $\nu > \nu_c$ the motion, in any event, will be completely in the nonlinear regime and stick slip dynamics may be observed.

Now a complex evolution is seen in F_t. As an example, in Fig. 5(a) we show the temporal evolution for F_t where we use the parameters $N = 200$, $\nu_c = 1$, and $\nu = 5$. Here F_t does not seem to be globally periodic.

For the BK model, when only the linear part of the friction force is felt, the chain behaves as a single block, as discussed in subsection (a). The motion for the center-of-mass is governed by Eq. (16) with $\Phi_\circ = 1$. From there we get $\dot{W} = \nu[1 - \cos \tau]$. Consequently, the maximum velocity attained by the blocks is given by $\dot{W}_{max} = 2\nu$. From here we find that when $\nu \geq \nu_c/2$ the nonlinear part of the friction force given by Eq. (18) will be felt, and the motion can be via stick slip.

We show in Fig. 5(b) an example for the temporal evolution of F_{BK}. The parameters are $N = 500$, $l^2 = 50$, $\nu_c = 1$, $b = 0.3$ and $\nu = 5$. Since $\nu > \nu_c$ the motion is nonlinear. We see a roughly periodic function with variable amplitude. The frequency here is one, which is the natural frequency of the center-of-mass motion, as given by Eq. (15).

We plot in Fig. 5(c) the mesoscopic friction force Φ (solid line) and the macroscopic forces $\overline{F}$ for the train model (circles) and for the BK model (squares) as functions of the pulling velocity. The respective parameters are the same as in Fig. 5(a) and 5(b). For the BK model there is a good agreement between the mesoscopic and macroscopic force. By rigidly shifting the mesoscopic force by a given factor, we will be able to make the curves practically coincide in the logarithmic region. The curves also coincide at the plateaux, where we have $\Phi_\circ = 1$ and $\Phi_\circ = 0$, as expected. But in this case no shift of the mesoscopic force is necessary. We have studied other regions of the parameter space and also found a good agreement between $\overline{F}_{BK}$ and $\Phi_\circ$.

The comparison between the macroscopic and mesoscopic force for the train model with this friction force shows, however, that the two curves do not present the same behavior in the logarithmic regime. This is clearly seen in Fig. 5(c) where the results for this model are represented by circles. In the region where the dynamics is linear the numerical results agree with the analytical ones. At $\nu = \nu_c/2$ we see the expected transitions to the nonlinear regime. When the motion becomes nonlinear there is a jump in $\overline{F}_t$ and the macroscopic force becomes much smaller than the mesoscopic one.

So, for the friction force proposed by Dietrich and Scholz we find a good agreement between the macroscopic and mesoscopic friction forces only for the BK model, so that the friction law seems to be scale invariant. For the train model there is no agreement between the two forces.

(c) Carlson-Langer friction force

Carlson and Langer[8] have studied the dynamics of the BK model using the velocity

weakening friction force given by

$$\Phi(\dot{U}/\nu_c) = \begin{cases} 1, & \text{if } \dot{U} = 0; \\ [1 + \dot{U}/\nu_c]^{-1}, & \text{if } \dot{U} > 0. \end{cases} \tag{19}$$

The train model with the friction force given by Eq. (19) was investigated in Ref. [9]. It was found that it presents slipping events of several sizes, and the distribution of event sizes was studied in detail.

The dynamics of the BK model, when governed by a friction force given by Eq. (19), has been investigated extensively in recent publications[8, 10, 11]. It has been shown that when ν is small there are basically two kinds of slipping events in the chain, which are the small and the large events. The small events displace a small number of blocks and they never attain velocities greater than ν_c. The small events, although numerous, relax almost no elastic energy, and they disappear in the continuum limit[11]. The small events also disappear for large ν. The big events displace a large number of blocks ($n \approx 4l^2\nu_c \ln[4l^2/\nu]$, as found in [8]) and they relax almost the total stress accumulated in the chain. They occur in a nearly periodic way.

In Figs. 6(a) and 6(b) we see temporal evolutions of the macroscopic force for train and BK models, respectively. We have taken $\nu_c = 1$ and $\nu = 1$. For the train model we see fluctuations of several sizes, with the largest ones representing the situation where all blocks of the chain are displaced. For the BK model the periodic character of the center-of-mass motion is also apparent here. The frequency is one. In these simulations we have used $N = 200$ for the train model, $N = 500$ and $l^2 = 50$ for the BK model.

The macroscopic force as a function of the pulling speed is shown in Fig. 6(c). We see that for the train model (circles) $\overline{F}_t$ is much smaller than the mesoscopic force (solid line) . We have studied other regions of the parameters space and no fixed line was found for the macroscopic and mesoscopic behavior in the train model when this friction force is used.

For the BK model (squares) we see a very good agreement between the mesoscopic and macroscopic forces. Therefore, for these parameter values, the macroscopic behavior is consistent with the mesoscopic one. The results shown are insensitive to variations in l^2, but present sensitivity on ν_c for small pulling speeds ($\nu \lesssim \nu_c$). If we decrease the characteristic velocity, that is, for $\nu_c < 1$ we see that the macroscopic force is much smaller than the mesoscopic one for small velocities, whereas the macroscopic force is significantly larger than the mesoscopic force if $\nu_c > 1$. For larger pulling velocities ($\nu \gtrsim \nu_c$) we always find good agreement between mesoscopic and macroscopic forces.

It has been shown [10] that the BK model presents a dynamic phase transition when $\nu_c = 1$. For characteristic velocities greater than this critical value the system moves in a continuous way. When $\nu_c < 1$ the motion is via stick slip. Thus, we see an excellent

agreement between the macroscopic and mesoscopic forces (for all values of ν) only at the critical point $\nu_c = 1$, where the motion changes its character. When the stick slip motion occurs ($\nu_c < 1$), much less force is necessary to displace the chain and one has a significant discrepancy between the macroscopic and mesoscopic friction forces. So, the stick slip motion seems to be an intelligent way the system uses to move with less effort.

In summary, the friction force suggested by Carlson and Langer has the same behavior in the macroscopic and mesoscopic levels only for the BK model at the critical velocity $\nu_c = 1$. A self-consistent relation between the macroscopic and mesoscopic forces is not found in the train model.

(d) Schmittbuhl-Vilotte-Roux force

Schmittbuhl *et al.* [6] recently investigated the average friction force on a rigid block that slides on a self-affine surface. Taking into consideration that the block jumps on ballistic trajectories, they found that for large velocities the friction force appears to be proportional to v^{-2}. For low velocities they found that the apparent friction coefficient is constant. The crossover between the two regimes is determined by a characteristic velocity. The equation we consider for this friction force is given by

$$\Phi(\dot{U}/\nu_c) = \begin{cases} 1, & \text{if } \dot{U} < \nu_c; \\ (\dot{U}/\nu_c)^{-2}, & \text{if } \dot{U} \geq \nu_c, \end{cases} \tag{20}$$

where we have introduced a cutoff at $\dot{U} = \nu_c$. When the chain experiences only the linear part of the friction force, that is, when no block attains velocity equal or greater than ν_c, an analytic solution is for the motion of the center-of-mass is possible for both models, as we have seen in subsection (b). There we have shown that the nonlinear part of the friction force is felt when $\nu \geq \nu_c/2$. In the train model there is an intermediate regime, which occurs when $\nu_c/2 \leq \nu \leq \nu_c$. In this regime F_t increases monotonically until the last block of the chain is displaced. Then, a sharp drop in F_t seen. After this, the macroscopic force starts to increase again, and the process repeats. For $\nu > \nu_c$ the motion of the train model becomes fully nonlinear and stick slip behavior can be observed.

We show a temporal evolution of F_t with $N = 200$, $\nu_c = 1$ and $\nu = 1.5$ in Fig. 7(a) and of F_{BK} in Fig. 7(b) with $N = 500$, $l^2 = 50$, $\nu_c = 1$ and $\nu = 1.5$. Again, we see a complex evolution for F in the train model with the largest jumps representing displacement of all blocks. For the BK model F presents a nearly periodic behavior with variable amplitude and frequency roughly equal to one.

The results of our simulations for the dependence of $\overline{F}$ with ν are shown in Fig. 7(c). For the train model (circles) we see a sharp drop in the macroscopic friction force when the motion becomes nonlinear, i.e., when $\nu \geq \nu_c/2$. For this model we find that asymptotically (for large ν) the macroscopic and mesoscopic (solid line) forces seem to present the same

behavior. It is difficult the study of $\overline{F}_t$ for much larger velocities as the ones we investigated because the statistical errors get very large and become of the order of $\overline{F}_t$.

For the BK model we see that the macroscopic force (circles) coincides with the mesoscopic one only in the plateaux region, where $\Phi = 1$, and the dynamics is fully linear. There is also a single point where the curves cross each other. We conclude that the friction force suggested in Ref. [6] does not give the same behavior for the macroscopic and mesoscopic levels in the BK model. However, note that in this case the macroscopic force has also a power law dependence on ν, but the exponent is clearly different from the exponent of the mesoscopic force.

In conclusion, we have studied the scale invariance of various proposed friction laws on two spring-block models of Burridge and Knopoff. We find that the classical Coulomb law, namely no velocity dependence of the friction force for finite velocity, is always invariant under a change from the mesoscopic to the macroscopic scale. The friction law proposed by Dietrich and Scholz are only invariant if the model with an upper bar is considered. The friction law used by Carlson and Langer is only invariant at the critical velocity. The relation recently suggested by Schmittbuhl *et al.* seems to be scale invariant for large pulling velocities in the train model. However, more precise simulations would be required for a definitive assessment.

Since the self-affinity of rough surfaces has been extensively documented, in particular for rocks the scale invariance seems to be an important condition for the force law. It would, in fact, be better to formulate a renormalization procedure for which the scale-invariant law would be "fixed points". This might allow to make a direct connection between the scaling of the friction force in terms of the roughness exponent. Finally, would it be also important to go to the microscopic scale by considering the geometrical hindrances represented by the asperities on rough surfaces.

ACKNOWLEDGMENTS

REFERENCES

1. C. H. Scholz, *The Mechanics of Earthquakes and Fauting*, Cambridge Univ. Press, New York, 1990.
2. J. Dietrich, *Pure Appl. Geophys.* **116**, 790 (1978).
3. C. H. Scholz and T. Engelder, *Int. J. Rock Mech. Min. Sci.* **13**, 149 (1972).

4. W. F. Brace and J. D. Byerlee, *Science* **153**, 990 (1966).

5. T. Pöschel and H. Herrmann, *Physica A* **198**, 441 (1993).

6. J. Schmittbuhl, J-P. Vilotte and S. Roux, to appear in *J. de Physique*.

7. R. Burridge and L. Knopoff, *Bull. Seismol. Soc. Am.* **57**, 341 (1967).

8. J. M. Carlson and J. S. Langer, *Phys. Rev. Lett.* **62**, 2632 (1989); *Phys. Rev. A* **40**, 6470 (1989).

9. M. de Sousa Vieira, *Phys. Rev. A* **46**, 6288 (1992).

10. G. Vasconcelos, M. de Sousa Vieira and S. R. Nagel, *Physica A* **191**, 69 (1992); M. de Sousa Vieira, G. Vasconcelos and S. R. Nagel, *Phys. Rev. E* **47**, R2221 (1993).

11. J. Schmittbuhl, J-P. Vilotte and S. Roux, *Europhys. Lett.* **21**, 375 (1993).

FIGURE CAPTIONS

Fig. 1. (a) Train model, which consists of a chain of blocks connected by linear springs. The blocks are on a flat surface and the first one is pulled with constant velocity. (b) BK model where each block is connected to a driving bar and to the neighboring blocks.

Fig. 2. (a) Evolution in phase space of a one-block system for different initial conditions. The central dot represents the trivial solution in which the block has constant velocity equal to the pulling speed $\nu = 0.1$. The ellipses have initial conditions given by $U(\tau = 0) = -\Phi_\circ = -0.8$ and $\dot{U}(\tau = 0) = 0.05, 0.025, 0.01, 0$. The initial velocities decrease in the outward direction. (b) Evolution in phase space of the first block in a two-block train system with $\Phi_\circ = 0.8$ for $\nu = 0.1$ giving a period two orbit. (c) The same as (b) with $\nu = 1$, which seems to result in a quasi-periodic motion.

Fig. 3 (a) Temporal evolution of the BK model with $N = 100$, $\Phi_\circ = 0.8$, $\nu = 0.01$ with (a) $l = 0$ and (b) $l^2 = 50$. The diagrams show the block number j versus τ. A dot in the figures means $\dot{U}_j > 0$.

Fig. 4. (a) Temporal evolution of the applied force F_t for the train model and (b) of F_{BK} associated with the BK model. The mesoscopic force is given by the Coulomb law with $\Phi_\circ = 0.8$. The parameters are $N = 200$ and $\nu = 5$ for the train model and $N = 500$, $l^2 = 50$, $\nu = 5$ for the BK model. (c) Mesoscopic force $\Phi_\circ$ (solid line) and macroscopic forces $\overline{F}$ versus the pulling velocity for the train model (circles) and the BK model (squares). With exception of ν, the parameter values are the same as in (a). The average macroscopic force was calculated in the train model for 1,000,000 time steps using intervals of $\delta\tau = 0.02$. For the BK model the temporal average was calculated for 200,000 iterations using time intervals of $\delta\tau = 0.01$.

Fig. 5. (a), (b) and (c), the same as in Fig. 4(a), 4(b) and 4(c), respectively, for the friction force given by Eq. (18), with $\nu_c = 1$, $b = 0.3$, and with $\nu = 5$ in (a) and (b). In Fig. 5(c) and 7(c) we needed to increase the number of iteration steps (in relation to Fig. 4(a)) in order to decrease the statistical erros associated with large pulling velocities.

Fig. 6. (a), (b), and (c), the same as in Fig. 4(a), 4(b), and 4(c), respectively, for the friction force given by Eq. (19), with $\nu_c = 1$ and with $\nu = 1$ in (a) and (b). The mesoscopic force here has been rigidly shifted by a constant factor in order to make it coincide with the macroscopic force of the BK model.

Fig. 7. (a), (b) and (c), the same as in Fig. 4(a), 4(b) and 4(c), respectively, for the friction force given by Eq. (20), with $\nu_c = 1$ and with $\nu = 1.5$ in (a) and (b).